5478: **Proposed by D. M. Bătinetu-Giurgiu**, **Matei Basarab National College**, **Bucharest**, **Romania**

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Compute:

$$\int_0^{\pi/2} \cos^2 x\left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right)dx.$$

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Let $I := \int_0^{\pi/2} \cos^2 x\left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right)dx.$

Since

$$I = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2} - x\right)\left(\cos\left(\frac{\pi}{2} - x\right)\sin^2\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2} - x\right)\right) + \sin\left(\frac{\pi}{2} - x\right)\sin^2\left(\frac{\pi}{2}\cos\left(\frac{\pi}{2} - x\right)\right)\right)dx$$

then denoting $t := \frac{\pi}{2} - x$ we obtain $dx = -dt$ and

$I = \int_{\pi/2}^0 \sin^2 t\left(\cos(t)\sin^2\left(\frac{\pi}{2}\sin(t)\right) + \sin(t)\sin^2\left(\frac{\pi}{2}\cos(t)\right)\right)(-dt) =$

$\int_0^{\pi/2} \sin^2 t\left(\cos(t)\sin^2\left(\frac{\pi}{2}\sin(t)\right) + \sin(t)\sin^2\left(\frac{\pi}{2}\cos(t)\right)\right)dt.$

Since $\cos^2 x\left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right) +$

$\sin^2 t\left(\cos x \sin^2\left(\frac{\pi}{2}\sin x\right) + \sin(t)\sin^2\left(\frac{\pi}{2}\cos x\right)\right) =$

$\left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right)(\cos^2 x + \sin^2 x) =$

$\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos(t)\sin^2\left(\frac{\pi}{2}\sin(t)\right)$ then

$2I = \int_0^{\pi/2}\left(\sin x \sin^2\left(\frac{\pi}{2}\cos x\right) + \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)\right)dx =$

$\int_0^{\pi/2} \sin x \sin^2\left(\frac{\pi}{2}\cos x\right)dx + \int_0^{\pi/2} \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)dx.$

Noting that $\int_0^{\pi/2} \sin x \sin^2\left(\frac{\pi}{2}\cos x\right)dx = \left[u := \frac{\pi}{2}\cos x, du = -\frac{\pi}{2}\sin x dx\right] =$

$-\frac{2}{\pi}\int_{\pi/2}^0 \sin^2 u(-du) = \frac{2}{\pi}\int_0^{\pi/2} \sin^2 u du = \frac{1}{\pi}\int_0^{\pi/2}(1 - \cos 2u)du = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}$

and $\int_0^{\pi/2} \cos x \sin^2\left(\frac{\pi}{2}\sin x\right)dx = \left[u := \frac{\pi}{2}\sin x, du = \frac{\pi}{2}\cos x dx\right] = \frac{2}{\pi}\int_0^{\pi/2} \sin^2 u du = \frac{1}{2}$

we obtain $2I = \frac{1}{2} + \frac{1}{2} = 1 \Leftrightarrow I = \frac{1}{2}.$